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Two Extremal Problems for Polynomials with an Interior Constraint

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1. INTRODUCTION

In 1976 Lorentz [6] presented some new results and posed some open questions concerning polynomials constrained to have a (possibly) high order zero at one endpoint of an interval. In particular on the interval [0, 1], the so called "incomplete" polynomials

$$x^s \sum_{i=0}^{m} a_i x^i \tag{1.1}$$

have been investigated extensively [1, 3, 4, 6, 7, 9–14]. Generalizing this notion of polynomials with *endpoint constraints*, several authors [5, 9] have studied polynomials of the form

$$(x-1)^{s_1} (x+1)^{s_2} \sum_{i=0}^{m} a_i x^i, \tag{1.2}$$

constrained at *both* endpoints of the interval [−1, 1]. The central theme in these early investigations has been to examine a family of constrained polynomials, of arbitrarily large degree, with zero of prescribed order at one or both endpoints. Some of the results obtained thus far concern the uniform approximation of continuous functions [1, 3, 6, 13], growth estimates [4, 5, 10, 11], and the distributions of zeros [9].

Quite naturally, analogous questions arise for polynomials possessing an *interior constraint* [9], that is, for polynomials on the intervals [−1, 1] having the form

$$(x-\lambda)^s \sum_{i=0}^{m} a_i x^i, \qquad -1 < \lambda < 1. \tag{1.3}$$

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While results regarding the special case $\lambda = 0$ follow from single endpoint considerations, the skewed cases have been absent from the literature. In this paper we investigate two distinct but related extremal problems posed for the polynomials of (1.3).

The outline of this paper is as follows: In Section 2 we introduce some needed notation and state the two extremal problems. We study in Section 3 the extremal polynomials associated with Problem I. We state and prove our main result in Section 4, concerning the extremal polynomials solving Problem II.

2. Notation and Extremal Problems

As usual, for each nonnegative integer m we let π_m denote the collection of real polynomials of degree at most m. For each pair of nonnegative integers s and m, we define

$$\pi_{s,m}(\lambda) := \{(x-\lambda)^s q_m(x) : q_m \in \pi_m\}, \tag{2.1}$$

where λ is a real number in $[-1, 1]$. Next let f be any real and continuous function defined on the interval $[-1, 1]$. We set

$$\|f\|_{[-1,1]} := \max\{|f(x)| : x \in [-1, 1]\}. \tag{2.2}$$

The collection

$$D := \{\xi \in [-1, 1] : |f(\xi)| = \|f\|_{[-1,1]}\} \tag{2.3}$$

is called the *set of extreme points of f*. Next, for each integer $m \geqslant 2$, let $\xi_1 < \xi_2 < \cdots < \xi_m$ be a subset of D with the property that

$$f(\xi_i) + f(\xi_{i+1}) = 0, \qquad i = 1, 2, \ldots, m-1. \tag{2.4}$$

Such a subcollection is called an *alternation set of f of length m.*

We now state our first extremal problem

Problem I. For each real number λ in $[-1, 1]$ and for each pair of nonnegative integers s and m, determine

$$E_{s,m}(\lambda) := \min \left\{ \left\| (x-\lambda)^s \prod_{i=1}^{m} (x - \alpha_i) \right\|_{[-1,1]} : \alpha_i \in \mathbb{R},\ i = 1, 2, \ldots, m \right\} \tag{2.5}$$

(where if $m = 0$, we take $\prod_{i=1}^{m} (x - \alpha_i) \equiv 1$).

Obviously the "free" real zeros of this extremal problem, that is, the $\alpha_1, \alpha_2, \ldots, \alpha_m$, are completely arbitrary. In Problem II, however, each is confined to the interval $[\lambda, 1]$.

Problem II. For each real number λ in $[-1, 1]$ and for each pair of nonnegative integers s and m, determine

$$e_{s,m}(\lambda) := \min \left\{ \left\| (x-\lambda)^s \prod_{i=1}^{m} (x-\alpha_i) \right\|_{[-1,1]} : \lambda \leqslant \alpha_i \leqslant 1,\ i = 1, 2, ..., m \right\} \quad (2.6)$$

(where if $m = 0$, we take $\prod_{i=1}^{m} (x-\alpha_i) \equiv 1$).

We remark that with the restriction of α_i $(i = 1, 2, ..., m)$ to the real numbers, Problem I is a nonlinear extremal problem. If this restriction is omitted, however, then Problem I becomes a *linear* weighted Chebyshev problem. Unique polynomial solutions exist for this linearized problem and since in this setting they are known to have all real zeros, there exist unique monic polynomials minimizing (2.5). These polynomials will each be denoted by $P_{s,m}^{(\lambda)}(x)$ and hence

$$\|P_{s,m}^{(\lambda)}\|_{[-1,1]} = E_{s,m}(\lambda). \quad (2.7)$$

Problem II is also a *nonlinear* extremal problem. It is easy to see that extremal polynomials exist for this latter problem since the set $[\lambda, 1]^m$ is compact in $\mathbb{R}^m$. We shall show in fact that these extremal polynomials are unique, to be denoted by $T_{s,m}^{(\lambda)}(x)$. Thus

$$\|T_{s,m}^{(\lambda)}\|_{[-1,1]} = e_{s,m}(\lambda). \quad (2.8)$$

Finally, we shall show that Problems I and II are related in the following way: For each choice of the three numbers s, m, and λ, there exists a unique integer $k = 0, 1, ..., m$ for which

$$T_{s,m}^{(\lambda)}(x) = P_{s+k,m-k}^{(\lambda)}(x); \quad (2.9a)$$

$$e_{s,m}(\lambda) = E_{s+k,m-k}(\lambda). \quad (2.9b)$$

3. Extremal Polynomials for Problem I

In this section we study the extremal polynomials solving Problem I. In addition we detail certain properties of these polynomials which will facilitate the results of Section 4.

Theorem 3.1. *For each real number λ in $[-1, 1]$ and for each pair of nonnegative integers s and m, there exists a unique monic polynomial of precise degree $n := s + m$*

$$P_{s,m}^{(\lambda)}(x) = (x-\lambda)^s\, p_{s,m}^{(\lambda)}(x) \quad (3.1)$$

satisfying (2.7). Moreover, for $m \geqslant 1$, the function $|x-\lambda|^s p_{s,m}^{(\lambda)}(x)$ has a (not necessarily unique) alternation set of precisely $m+1$ distinct points

$$-1 \leqslant \xi_0^{(s,m,\lambda)} < \xi_1^{(s,m,\lambda)} < \cdots < \xi_m^{(s,m,\lambda)} \leqslant 1 \tag{3.2}$$

for which

$$|\xi_i^{(s,m,\lambda)} - \lambda|^s p_{s,m}^{(\lambda)}(\xi_i^{(s,m,\lambda)}) = (-1)^{m-i} E_{s,m}(\lambda), \qquad i = 0, 1, ..., m. \tag{3.3}$$

Conversely, let $p(x)$ be a monic polynomial of degree at most $m \geqslant 1$, and let the function $|x-\lambda|^s p(x)$ have an alternation set of at least $m+1$ points in $[-1, 1]$. Then

$$p(x) \equiv p_{s,m}^{(\lambda)}(x). \tag{3.4}$$

Since the arguments needed for this result are rather standard, we only provide a sketch of the proof. Suppose the $a_i (i = 1, 2, ..., m)$ in Problem I are not constrained to be real. Then Problem I becomes a linear weighted Chebyshev problem. In this case it is known (cf. Walsh [15, p. 363]) that there exist unique monic polynomials minimizing (2.5). In addition, since the interval of interest to us here is $[-1, 1]$, the zeros of these extremal polynomials are real in $[-1, 1]$, making them at once solutions to the nonlinear Problem I as orginally posed. For the equioscillation characterization of $P_{s,m}^{(\lambda)}(x)$ we refer the reader to Meinardus [8].

It was suggested in the preceding theorem that the alternation set associated with the function $|x-\lambda|^s p_{s,m}^{(\lambda)}(x)$ need not be unique. This is true for certain choices of the parameter λ. To produce an example, we study the incomplete polynomial $P_{s,m}^{(-1)}(x)$. Since this polynomial has all of its zeros in $[-1, 1]$, it is monotone for $x \leqslant -1$. Consequently, for each pair of nonnegative intergers s and m, not both zero, there exists a unique real number $r_{s,m} \geqslant 1$ for which

$$|P_{s,m}^{(-1)}(-r_{s,m})| = E_{s,m}(-1) \tag{3.5}$$

(cf. [11]). Next, map the interval $[-r_{s,m}, 1]$ linearly to the interval $[-1, 1]$ and define $\lambda_{s,m}$ to be the image of -1, that is,

$$\lambda_{s,m} := (r_{s,m} - 3)/(r_{s,m} + 1). \tag{3.6}$$

Now for $m \geqslant 1$, according to Theorem 3.1, the polynomial $P_{s,m}^{(-1)}(x)$ has $m+1$ alternation points in $[-1, 1]$. By instituting a change of variable, this equioscillation property is preserved by the polynomial

$$P_{s,m}^{(-1)}(2[(x-\lambda)/(1-\lambda)] - 1),$$

for each λ in $[-1, \lambda_{s,m}]$. Thus by the second half of Theorem 3.1, after normalization,

$$P_{s,m}^{(\lambda)}(x) = [(1-\lambda)/2]^{s+m} P_{s,m}^{(-1)}(2[(x-\lambda)/(1-\lambda)]-1); \tag{3.7a}$$

$$E_{s,m}(\lambda) = [(1-\lambda)/2]^{s+m} E_{s,m}(-1), \tag{3.7b}$$

for each λ in $[-1, \lambda_{s,m}]$. In particular, for positive integers s and m, the function

$$|x - \lambda_{s,m}|^s p_{s,m}^{(\lambda_{s,m})}(x)$$

has *two* alternation sets, each of length $m+1$. This behavior is illustrated in Fig. 1.

We remark that while we do not have a general representation for $\lambda_{s,m}$, numerical estimates are readily available.

Not only do the parameter values $\lambda_{s,m}$ provide instances for the nonunicity of alternation sets for the functions $|x-\lambda|^s p_{s,m}^{(\lambda)}(x)$, but they play a key role in the solution of Problem II. Thus in the remainder of this section we shall develop certain relationships between the numbers $\lambda_{s,m}$ and the polynomials $P_{s,m}^{(\lambda)}(x)$. First we require

LEMMA 3.2 ([5, 11]). *Let $Q(x)$ be an arbitrary polynomial from $\pi_{s,m}(-1)$, not a scalar multiple of $P_{s,m}^{(-1)}(x)$. Then for each $|x|>1$,*

$$|Q(x)|/\|Q\|_{[-1,1]} < |P_{s,m}^{(-1)}(x)|/E_{s,m}(-1). \tag{3.8}$$

We make use of this lemma in the proof of

THEOREM 3.3. *For each pair of integers $s \geqslant 0$ and $m \geqslant 1$,*

$$-1 \leqslant \lambda_{s,m} < \lambda_{s+1,m-1} \leqslant 0. \tag{3.9}$$

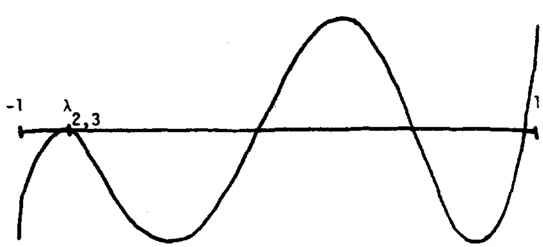

FIG. 1. $P_{2,3}^{(\lambda_{2,3})}(x)$.

Proof. Since $\pi_{s+1,m-1}(-1) \subset \pi_{s,m}(-1)$, we can take $Q(x) = P^{(-1)}_{s+1,m-1}(x)$ in Lemma 3.2. Then, when $x = -r_{s,m}$, inequality (3.8) together with (3.5) yields

$$|P^{(-1)}_{s+1,m-1}(-r_{s,m})| < E_{s+1,m-1}(-1). \tag{3.10}$$

Since $P^{(-1)}_{s+1,m-1}(x)$ is monotone and nonzero for $x < -1$ and since $|P_{s+1,m-1}(-r_{s+1,m-1})| = E_{s+1,m-1}(-1)$, (3.10) implies that $-r_{s+1,m-1} < -r_{s,m}$. From the definition of $\lambda_{s,m}$ in (3.6), it follows that $\lambda_{s,m} < \lambda_{s+1,m-1}$. Finally, it is a simple exercise to verify that $r_{n,0} = 3$ and $r_{0,n} = 1$, for each $n \geqslant 1$, from which the upper and lower bounds of (3.9) easily follow.

In the last theorem of this section we determine the location of the least nontrivial zero of $P^{(\lambda)}_{s,m}(x)$ with respect to the parameter λ. For the proof we shall need the continuity result of

LEMMA 3.4 ([2]). *Let $\{\mu_l\}_{l=1}^{\infty}$ be an infinite sequence of real numbers in the interval $[-1, 1]$ and suppose*

$$\lim_{l \to \infty} \mu_l = \mu. \tag{3.11}$$

Then for each pair of nonnegative integers s and m,

$$\lim_{l \to \infty} P^{(\mu_l)}_{s,m}(z) = P^{(\mu)}_{s,m}(z), \qquad \text{for all} \quad z \text{ in } \mathbb{C}; \tag{3.12a}$$

$$\lim_{l \to \infty} E_{s,m}(\mu_l) = E_{s,m}(\mu). \tag{3.12b}$$

Furthermore, this convergence is uniform on compact subsets of $\mathbb{C}$.

THEOREM 3.5. *For each pair of integers $s \geqslant 0$ and $m \geqslant 1$, write*

$$P^{(\lambda)}_{s,m}(x) = (x-\lambda)^s \prod_{i=1}^{m} (x - \alpha_i^{(s,m,\lambda)}), \tag{3.13}$$

where

$$\alpha_1^{(s,m,\lambda)} < \alpha_2^{(s,m,\lambda)} < \cdots < \alpha_m^{(s,m,\lambda)}. \tag{3.14}$$

Then

$$\lambda < \alpha_1^{(s,m,\lambda)} \qquad \text{for} \quad -1 \leqslant \lambda < \lambda_{s+1,m-1}, \tag{3.15a}$$

$$\lambda = \alpha_1^{(s,m,\lambda)} \qquad \text{for} \quad \lambda = \lambda_{s+1,m-1}, \tag{3.15b}$$

$$\lambda > \alpha_1^{(s,m,\lambda)} \qquad \text{for} \quad \lambda_{s+1,m-1} < \lambda \leqslant 1. \tag{3.15c}$$

That the zeros of $P_{s,m}^{(\lambda)}(x)$ may be written as in (3.14) follows as a consequence of (3.2) and (3.3). Before we proceed with the proof of Theorem 3.5, we find it convenient to state the result of Lemma 3.6 separately.

LEMMA 3.6. *Let* s, m, λ, *and* $\alpha_1^{(s,m,\lambda)}$ *be as in Theorem* 3.5. *Then* $\lambda = \alpha_1^{(s,m,\lambda)}$ *implies that* $\lambda = \lambda_{s+1,m-1}$.

Proof of Lemma 3.6. Assuming that $\lambda = \alpha_1^{(s,m,\lambda)}$, it follows that

$$P_{s,m}^{(\lambda)}(x) = P_{s+1,m-1}^{(\lambda)}(x). \tag{3.16}$$

Based upon properties of $P_{s,m}^{(\lambda)}(x)$ already discussed, the polynomial $P_{s+1,m-1}^{(\lambda)}(x)$ must attain its extreme values $m+1$ times in $[-1, 1]$. Since $m-1$ is the maximum number of critical points for $P_{s+1,m-1}^{(\lambda)}(x)$ in $(-1, 1) - \{\lambda\}$, both $x=-1$ and $x=1$ must be among these extreme points. Moreover, since the "free" zeros of $P_{s+1,m-1}^{(\lambda)}(x)$ are simple in $(\lambda, 1)$, this polynomial also has an alternation set of m points *in the interval* $[\lambda, 1]$.

To simplify the notation in this proof we let $\mu = \lambda_{s+1,m-1}$. Now from the definition of $\lambda_{s,m}$ in (3.6), these same properties may be attributed to the polynomial

$$P_{s+1,m-1}^{(\mu)}(x) = [(1-\mu)/2]^{s+m}\, P_{s+1,m-1}^{(-1)}(2[(x-\mu)/(1-\mu)]-1). \tag{3.17}$$

That is, $P_{s+1,m-1}^{(\mu)}(x)$ attains its extreme values $m+1$ times in $[-1, 1]$, including $x=-1$ and $x=1$, and has an alternation set of m points in the interval $[\mu, 1]$.

We claim that $\lambda = \mu$ and hence $P_{s+1,m-1}^{(\lambda)} = P_{s+1,m-1}^{(\mu)}(x)$. To this end we define

$$Q(x) := [(1-\mu)/(1-\lambda)]^{s+m}\, P_{s+1,m-1}^{(\lambda)}([(1-\lambda)/(1-\mu)](x-\mu)+\lambda); \tag{3.18}$$

$$R(x) := [(1-\lambda)/(1-\mu)]^{s+m}\, P_{s+1,m-1}^{(\mu)}([(1-\mu)/(1-\lambda)](x-\lambda)+\mu). \tag{3.19}$$

It is easy to see that $Q(x)$ and $R(x)$ are monic polynomials in $\pi_{s+1,m-1}(\mu)$ and $\pi_{s+1,m-1}(\lambda)$, respectively. From our preceding remarks, if $\mu \leqslant \lambda$, then the monic polynomial $Q(x)$ has an alternation set of m points in the interval $[-1, 1]$. According to the converse statement in Theorem 3.1 then

$$Q(x) \equiv P_{s+1,m-1}^{(\mu)}(x). \tag{3.20}$$

Similarly, when $\lambda \leqslant \mu$, we have

$$R(x) \equiv P_{s+1,m-1}^{(\lambda)}(x). \tag{3.21}$$

In either case we may write

$$P_{s+1,m-1}^{(\lambda)}(x) = ((1-\lambda)/(1-\mu))^{s+m}\, P_{s+1,m-1}^{(\mu)}([(1-\mu)/(1-\lambda)](x-\lambda)+\mu). \tag{3.22}$$

Since $x = 1$ is an extreme point for both $P^{(\lambda)}_{s+1,m-1}(x)$ and $P^{(\mu)}_{s+1,m-1}(x)$, it follows from (3.22) that

$$E_{s+1,m-1}(\lambda) = ((1-\lambda)/(1-\mu))^{s+m}\, E_{s+1,m-1}(\mu). \tag{3.23}$$

Now suppose $\mu < \lambda$; then it is easy to see that

$$(1-\mu)(-1-\lambda)/(1-\lambda) + \mu < -1. \tag{3.24}$$

Recalling the fact that $|P^{(\mu)}_{s+1,m-1}(x)| > E_{s+1,m-1}(\mu)$ for all $x < -1$, we combine (3.22) and (3.24) to obtain

$$E_{s+1,m-1}(\lambda) > ((1-\lambda)/(1-\mu))^{s+m}\, E_{s+1,m-1}(\mu).$$

Since this is a clear contradiction to (3.23), $\mu \geqslant \lambda$. Similarly, if we suppose that $\mu > \lambda$, we again shall obtain a contradiction to (3.23). Consequently $\mu = \lambda$, which was to be proved.

We now continue with the

Proof of Theorem 3.5. For $-1 < \lambda < \lambda_{s,m}$, it is known (cf. (3.7a)) that

$$P^{(\lambda)}_{s,m}(x) = [(1-\lambda)/2]^{s+m}\, P^{(-1)}_{s,m}(2[(x-\lambda)/(1-\lambda)] - 1).$$

Clearly for these λ we have $\lambda < \alpha_1^{(s,m,\lambda)}$. Let Ω denote the collection of all such λ in the interval $(-1, 1)$, that is,

$$\Omega := \{\lambda \in (-1, 1) \colon \lambda < \alpha_1^{(s,m,\lambda)}\}. \tag{3.25}$$

While Ω is nonempty, we shall show further that it is open and is equal to $(-1, \lambda_{s+1,m-1})$.

To show that Ω is open, let $\lambda \in \Omega$. We shall show that λ is an interior point of Ω. For $\rho > 0$, set

$$C^i_\rho := \{z \in \mathbb{C} \colon |z - \alpha_i^{(s,m,\lambda)}| = \rho\}, \qquad i = 1, 2, \ldots, m, \tag{3.26}$$

and fix ρ sufficiently small so that $P^{(\lambda)}_{s,m}(x)$ has precisely one zero in each C^i_ρ, $i = 1, 2, \ldots, m$, and so that $\lambda < \alpha_1^{(s,m,\lambda)} - \rho$. Next let

$$\varepsilon := \min_{1 \leqslant i \leqslant m} \min\{|P^{(\lambda)}_{s,m}(z)| \colon z \in C^i_\rho\} > 0. \tag{3.27}$$

For this value of ε, we use Lemma 3.4 to determine $\delta > 0$ so that $|\mu - \lambda| < \delta$ implies

$$\|P^{(\mu)}_{s,m} - P^{(\lambda)}_{s,m}\|_{C^i_\rho} < \varepsilon, \qquad i = 1, 2, \ldots, m, \tag{3.28}$$

and

$$\mu \notin C_\rho^1. \tag{3.29}$$

Using (3.27) and (3.28), Rouché's Theorem implies that $P_{s,m}^{(\mu)}(x)$ has a simple (real) zero in each C_ρ^i, $i = 1, 2, \ldots, m$. This fact, together with (3.29), implies that $\mu < \alpha_1^{(s,m,\mu)}$ and hence $\mu \in \Omega$. Thus Ω is open.

Since Ω is the union of open intervals in $(-1, 1)$, we let I represent any one of these and define $\mu := \sup I$. We shall show that $\mu = \lambda_{s+1,m-1}$ and hence that $\Omega = (-1, \lambda_{s+1,m-1})$. Since $\mu \notin \Omega$, we have $\alpha_1^{(s,m,\mu)} \leqslant \mu$. If the strict inequality holds, then an argument such as that used to show that Ω is open can be used to construct a neighborhood about μ in which $\alpha_1^{(s,m,\lambda)} < \lambda$. Such a neighborhood, however, would have a nonempty intersection with Ω, which is impossible. As a result, $\mu = \alpha_1^{(s,m,\lambda)}$. But as a consequence of Lemma 3.6, this implies that $\mu = \lambda_{s+1,m-1}$, from which it follows that $\Omega = (-1, \lambda_{s+1,m-1})$. Thus for $-1 \leqslant \lambda < \lambda_{s+1,m-1}$, we have $\alpha_1^{(s,m,\lambda)} > \lambda$, proving (3.15a).

Next suppose $\lambda_{s+1,m-1} < \lambda \leqslant 1$. Since $\lambda \notin \Omega$, we have $\alpha_1^{(s,m,\lambda)} \leqslant \lambda$. Now if the strict inequality should hold, then, as indicated above, we can determine a neighborhood about $\lambda_{s+1,m-1}$ in which $\alpha_1^{(s,m,\lambda)} < \lambda$. As this contradicts the fact that $\Omega = (-1, \lambda_{s+1,m-1})$, we conclude that $\alpha_1^{(s,m,\lambda)} = \lambda$. The theorem is now completely proved.

4. Extremal Polynomials for Problem II

In this last section we state and prove the main result of this note concerning the existence and uniqueness of extremal polynomials $T_{s,m}^{(\lambda)}(x)$ solving the nonlinear extremal Problem II.

THEOREM 4.1. *For each real number λ in $[-1, 1]$ and for each pair of nonnegative integers s and m, there exists a unique monic polynomial of precise degree $n := s + m$*

$$T_{s,m}^{(\lambda)}(x) = (x-\lambda)^s \, t_{s,m}^{(\lambda)}(x), \tag{4.1}$$

with $t_{s,m}^{(\lambda)}(x)$ having all its zeros in the interval $[\lambda, 1]$ and satisfying (2.8). *For $m \geqslant 2$, define the intervals*

$$\begin{aligned} \Lambda_0^{(s,m)} &:= [-1, \lambda_{s+1,m-1}], \\ \Lambda_i^{(s,m)} &:= (\lambda_{s+i,m-i}, \lambda_{s+i+1,m-i-1}], \qquad i = 1, 2, \ldots, m-1, \\ \Lambda_m^{(s,m)} &:= (\lambda_{n,0}, 1] = (0, 1]. \end{aligned} \tag{4.2}$$

When $m=1$, *set* $\Lambda_0^{(s,m)} := [-1,0]$ *and* $\Lambda_1^{(s,m)} := (0,1]$. *Then for* $m \geqslant 1$,

$$T_{s,m}^{(\lambda)}(x) = P_{s+k,m-k}^{(\lambda)}(x), \tag{4.3a}$$

$$e_{s,m}(\lambda) = E_{s+k,m-k}(\lambda), \tag{4.3b}$$

for each λ *in* $\Lambda_k^{(s,m)}$, $k=0, 1,..., m$.

Before we prove Theorem 4.1, we present two lemmas which detail certain characteristics of the extremal polynomials of Problem II. The first of these is

LEMMA 4.2. *For each number* λ *in* $[-1,1]$ *and for each pair of integers* $s \geqslant 0$ *and* $m \geqslant 1$, *let* $T(x)$ *denote any extremal polynomial of Problem* II *so that* $\|T\|_{[-1,1]} = e_{s,m}(\lambda)$. *For a unique integer* $k = 0, 1,..., m$, *we can write*

$$T(x) = (x-\lambda)^{s+k}\, t(x), \tag{4.4}$$

where $t(x)$ *is monic in* π_{m-k} *and* $t(\lambda) \neq 0$. *If* $k < m$, *then there exist at least* $m-k+1$ *points* $\xi_0 < \xi_1 < \cdots < \xi_{m-k}$ *for which*

$$|\xi_i - \lambda|^{s+k}\, t(\xi_i) = (-1)^{m-k-i}\, e_{s,m}(\lambda), \qquad i = 0, 1,..., m-k. \tag{4.5}$$

Proof. Assume $k < m$. We first show that $t(x)$ has simple zeros in the interval $(\lambda, 1)$. For convenience we write

$$t(x) = \prod_{i=1}^{m-k} (x-\alpha_i), \tag{4.6}$$

where $\lambda < \alpha_1 \leqslant \cdots \leqslant \alpha_{m-k} \leqslant 1$. First suppose $\alpha_{m-k} = 1$. Then for small $\varepsilon > 0$ the polynomial

$$Q(x;\varepsilon) := T(x)((x-\alpha_{m-k}+\varepsilon)/(x-\alpha_{m-k})) \tag{4.7}$$

is a competitor of $T(x)$ and for ε sufficiently small, $\|Q\|_{[-1,1]} < e_{s,m}(\lambda)$, yielding a contradiction. Thus $\alpha_{m-k} < 1$. We next show that there can be no multiple zeros. Suppose $m-k \geqslant 2$ and let $j = 1, 2,..., m-k-1$ be any integer for which $\alpha_j = \alpha_{j+1}$. Then for ε small and positive

$$Q(x;\varepsilon) := T(x)((x-\alpha_j+\varepsilon)(x-\alpha_{j+1}-\varepsilon)/(x-\alpha_j)(x-\alpha_{j+1})) \tag{4.8}$$

is again a competitor of $T(x)$. For ε suitably small $\|Q\|_{[-1,1]} < e_{s,m}(\lambda)$, yielding a contradiction. Consequently each zero of $t(x)$ is simple in $(\lambda, 1)$.

We now show there exist at least $m-k+1$ extreme points for the polynomial $T(x)$ in $[-1,1]$. It is easy to see that $x=1$ is extreme for $T(x)$, or the polynomial of (4.7) would, for ε sufficiently small, provide a uniformly smaller extremal. Also, each of the $m-k-1$ critical points of

$T(x)$ in (α_1, α_{m-k}) are extreme points. If this were not true for the critical point in some interval (α_j, α_{j+1}), $j = 1, 2, \ldots, m-k-1$, then ε could be chosen sufficiently small in (4.8) so that $\|Q(\cdot\,;\varepsilon)\|_{[-1,1]} < e_{s,m}(\lambda)$, which is a contradiction. Finally, either $x = -1$ or the critical point in (λ, α_1) is extreme for $T(x)$. If not, for ε suitably small, the polynomial

$$Q((x;\varepsilon) := T(x)((x-\alpha_1-\varepsilon)/(x-\alpha_1))$$

can be made to satisfy $\|Q(\cdot\,;\varepsilon)\|_{[-1,1]} < e_{s,m}(\lambda)$, an impossibility. Thus $T(x)$ has at least $m-k+1$ extreme points in $[-1, 1]$.

We now argue that these extreme points outlined above provide an alternation set of length $m-k+1$ for the function $|x-\lambda|^{s+k}\, t(x)$. Let ξ_0 be any extreme point in $[-1, \alpha_1)$ and label the $m-k$ extrema in $(\alpha_1, 1]$ by $\xi_1 < \xi_2 < \cdots < \xi_{m-k} = 1$. We shall show that these points satisfy (4.5). First, since $T(x)$ is monic and monotone for $x > \alpha_{m-k}$,

$$|\xi_{m-k} - \lambda|^{s+k}\, t(\xi_{m-k}) = e_{s,m}(\lambda).$$

Next, since for each $j = 0, 1, \ldots, m-k-1$, the points ξ_j and ξ_{j+1} are separated by a single simple zero of $t(x)$, we have

$$t(\xi_j) \cdot t(\xi_{j+1}) < 0, \qquad j = 0, 1, \ldots, m-k-1.$$

This, together with the fact that

$$|T(\xi_i)| = e_{s,m}(\lambda), \qquad i = 0, 1, \ldots, m-k,$$

completes the proof of the lemma.

Next we prove

LEMMA 4.3. *For each real number λ in $[-1, 1]$ and each pair of integers $s \geqslant 0$ and $m \geqslant 1$, let $T(x)$ be any extremal polynomial solving Problem* II. *Thus $\|T\|_{[-1,1]} = e_{s,m}(\lambda)$. If $\lambda_{s+k,m-k} < \lambda \leqslant 1$, for $k = 1, 2, \ldots, m$, then*

$$T(x) = (x-\lambda)^{s+k}\, t(x), \tag{4.9}$$

where $t \in \pi_{m-k}$.

Proof. Let $k = 1$ in the lemma and write

$$T(x) = (x-\lambda)^s\, t(x),$$

where $t \in \pi_m$. We must show that $t(\lambda) = 0$ for $\lambda_{s+1,m-1} \leqslant \lambda \leqslant 1$. Suppose this is not the case and consider the functon $q(x)$ defined by

$$|x-\lambda|^s\, q(x) := |x-\lambda|^s\, (t(x) - p_{s,m}^{(\lambda)}(x)).$$

Since both $t(x)$ and $p_{s,m}^{(\lambda)}(x)$ are monic polynomials of precise degree m, $q \in \pi_{m-1}$. Furthermore, since $T(x)$ is a competitor of the unique extremal polynomial $P_{s,m}^{(\lambda)}(x)$ in Problem I, it follows that $e_{s,m}^{(\lambda)} > E_{s,m}(\lambda)$ or $T(x) \equiv P_{s,m}^{(\lambda)}(x)$. According to Theorem 3.5, the latter is impossible since $T(x)$ is zero free in $[-1, \lambda)$, while $P_{s,m}^{(\lambda)}(x)$ has at least a single zero in this interval. Thus $e_{s,m}(\lambda) > E_{s,m}(\lambda)$. Lemma 4.2 guarantees, however, that $|x-\lambda|^s\, t(x)$ has an alternation set of $m+1$ points, forcing m sign changes for the polynomial $q(x)$. As this implies $q(x) \equiv 0$ or $T(x) \equiv P_{s,m}^{(\lambda)}(x)$, an impossibility, we have shown that $t(\lambda) = 0$, proving the lemma when $k = 1$.

We now use induction on k. Suppose the lemma to be valid for $k = K < m$, and suppose for $\lambda_{s+K+1,m-K-1} < \lambda \leqslant 1$ that

$$T(x) = (x-\lambda)^{s+K}\, t(x),$$

where $t \in \pi_{m-K}$ and $t(\lambda) \neq 0$. Then define q by

$$|x-\lambda|^{s+K}\, q(x) := |x-\lambda|^{s+K}\, (t(x) - p_{s+K,m-K}^{(\lambda)}(x)).$$

Paralleling our argument for the case $k = 1$, we note that $q \varepsilon \pi_{m-K-1}$. Moreover, either $e_{s,m}(\lambda) > E_{s,m}(\lambda)$ or $T(x) \equiv P_{s+K,m-K}^{(\lambda)}(x)$, by the uniqueness of the extremal polynomial for Problem I. Since Theorem 3.5 rules out the latter possibility, we have $e_{s,m}(\lambda) > E_{s,m}(\lambda)$. Lemma 4.2, however, implies that $q(x)$ vanishes $m-K$ times in $[-1, 1]$, forcing it to be identically zero. But this means $T(x) \equiv P_{s+K,m-K}^{(\lambda)}(x)$, contradicting Theorem 3.5 and our assumption that $T(x)$ was an extremal polynomial for Problem II. Thus the lemma is valid for $k = K+1 \leqslant m$. This completes the proof.

Proof of Theorem 4.1. Since $T_{s,m}^{(\lambda)}(x) = (x-\lambda)^s$ when $m = 0$, we shall assume that $m \geqslant 1$. Let $\lambda \in \Lambda_k$ for some $k = 0, 1, \ldots, m$, and let $T(x)$ be any extremal polynomial solving Problem II. Then according to the definition of Λ_k and Lemma 4.3, we can write

$$T(x) = (x-\lambda)^{s+k}\, t(x), \tag{4.10}$$

where $t \in \pi_{n-k}$. Now since $\pi_{s+k,m-k}(\lambda) \subset \pi_{s,m}(\lambda)$, it is clear that

$$e_{s+k,m-k}(\lambda) \geqslant e_{s,m}(\lambda). \tag{4.11}$$

But because $T \in \pi_{s+k,m-k}(\lambda)$ and has its remaining zeros in $[\lambda, 1]$,

$$e_{s+k,m-k}(\lambda) \leqslant \|T\|_{[-1,1]} = e_{s,m}(\lambda). \tag{4.12}$$

Combining (4.11) and (4.12), it follows that

$$\|T\|_{[-1,1]} = e_{s+k,m-k}(\lambda). \tag{4.13}$$

We now observe that $T(x)$ and $P_{s,m}^{(\lambda)}(x)$ are competitors of one another. That is, according to Theorem 3.5 for $\lambda \in \Lambda_k$, $P_{s+k,m-k}^{(\lambda)}(x)$ has all of its "free" zeros in $[\lambda, 1]$ and hence

$$e_{s+k,m-k}(\lambda) \leqslant \|P_{s+k,m-k}^{(\lambda)}\|_{[-1,1]} = E_{s+k,m-k}(\lambda); \tag{4.14}$$

while comparing Problems I and II,

$$E_{s+k,m-k}(\lambda) \leqslant \|T\|_{[-1,1]} = e_{s+k,m-k}(\lambda). \tag{4.15}$$

Finally, combining (4.14) and (4.15), we conclude that $T(x) \equiv P_{s+k,m-k}^{(\lambda)}(x)$, according to the characterization criteria of the unique extremal polynomial for Problem I. This completes the proof of the theorem.

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